Tut 12 Revision

Rules for gradient: Product rule: $\nabla(fg) = f\nabla g + g\nabla f$

Quotient rule: $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Rules for divergence: Product rule: $\nabla\cdot(fA) = \nabla f\cdot A + f\nabla\cdot A$

Quotient rule: $\nabla\cdot\left(\frac{A}{g}\right) = \frac{g\nabla\cdot A - \nabla g\cdot A}{g^2}$

Rules for curl: Product rule: $\nabla\times(fA) = f\,\nabla\times A + \nabla f\times A$

Quotient rule: $\nabla\times\left(\frac{A}{g}\right) = \frac{g\nabla\times A - \nabla g\times A}{g^2}$

Chain rules: $A: \mathbb{R}^3 \to \mathbb{R}^3$ vector field

$F: \mathbb{R}^3 \to \mathbb{R}$ function

$\alpha: \mathbb{R} \to \mathbb{R}^3$ curve

$f: \mathbb{R} \to \mathbb{R}$

<u>Q1</u>: a) $\nabla(f\circ F) = (f'\circ F)\nabla F$

b) $(F\circ\alpha)' = (\nabla F\circ\alpha)\cdot\alpha'$

Ans: a) $\partial_x(f\circ F) = f'\circ F\,\partial_x F$

$\partial_y(f\circ F) = f\circ F\,\partial_y F \quad \Rightarrow \quad \nabla(f\circ F) = f'\circ F\,(\partial_x F, \partial_y F, \partial_z F)$

$\partial_z(f\circ F) = f\circ F\,\partial_z F \qquad\qquad = (f'\circ F)\,\nabla F$

b) The chain rule proved in lectures

Q2 Find f such that $\nabla f = 2x\vec{i} + 3y^2z\vec{j} + y^3\vec{k}$

Ans: $\partial_x f = 2x$ — ①

$\partial_y f = 3y^2z$ — ②

$\partial_z f = y^3$ — ③

① $\Rightarrow$ $f = x^2 + g(y,z)$

② $\Rightarrow$ $\partial_y g = 3y^2z \Rightarrow y^3z + h(z)$, or $f = x^2 + y^3z + h(z)$

③ $\Rightarrow$ $y^3 + \partial_z h = y^3 \Rightarrow \partial_z h = 0 \Rightarrow h = \text{Constant}$

$\Rightarrow f = x^2 + y^3z + \text{Constant}$

Stokes like thm in $\mathbb{R}^3$.

We can integrate over a curve, a surface or a (3-dim) Region.

$$\int_\alpha \vec{A} = \int_\alpha \vec{A}\, d\vec{\alpha} = \int_\alpha \vec{A}\cdot\vec{T}\, ds$$

$$\iint_S \vec{A} = \iint_S \vec{A}\, d\vec{S} = \iint_S \vec{A}\cdot\vec{n}\, d\sigma$$

$$\iiint_R f = \iiint_R f\, dv$$

$$\begin{array}{lccccccc} \mathbb{R}^3: & 0 & & 1 & & 2 & & 3 \\ & \text{Fn} & \xrightarrow{\text{grad}} & \text{VF} & \xrightarrow{\text{curl}} & \text{VF} & \xrightarrow{\text{div}} & \text{Fn} \end{array}$$

① Composition of two successive operators is zero.

② Stokes like thm

$$\int_{\partial R} \square \overset{+1 \text{ dim}}{=} \int_R \square$$

① $\operatorname{curl}(\operatorname{grad}(f)) = 0$

$\operatorname{div}(\operatorname{curl}(\vec{F})) = 0$

② $f(\gamma(b)) - f(\gamma(a)) = \int_\gamma \operatorname{grad}(f)$

$$\int_{\partial R} \vec{F} = \iint_R \operatorname{curl}(\vec{F})$$

$$\iint_{\partial D} \vec{F} = \iiint_D \operatorname{div}(\vec{F})$$

Tut 8

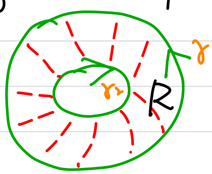
Green's thm: $\int_{\partial R} P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$

provided: ∂R is the disjoint union of simple closed curves.
(and R is contained in some open subset of $\mathbb{R}^2$ on which P, Q are smooth)

Special case: $\int_C x\,dy = -\int_C y\,dx = \int_R 1\,dx\,dy = \text{Area}(R)$

Rmk: Positively oriented means while you are traveling along the curve, R is on your left

Q3: Find, using Green's thm, the area of the region R bounded by the $x^2+y^2=4$ and $x^2+y^2=1$



Ans: Formula: $\text{Area}(R) = -\int_{\partial R} y\,dx$ (or $\int x\,dy$)

Step 1: Write done γ_1, γ_2

$\gamma_1(t) = (2\cos t, 2\sin t) \quad 0 \le t \le 2\pi$

$\gamma_2(t) = (\cos(-t), \sin(-t)) \quad 0 \le t \le 2\pi$

Step 2: $\text{Area} = \int_0^{2\pi} -2\sin t\, d\,2\cos t + \int_0^{2\pi} -\sin(-t)\, d\cos$:)

$= \int_0^{2\pi} 4\sin^2 t\,dt + \int_0^{2\pi} -\sin^2 t\,dt$

$= \int_0^{2\pi} 3\sin^2 t\,dt$

$= \frac{3}{2}\int_0^{2\pi} 1-\cos 2t\,dt = 3\pi$

Polar coordinates:

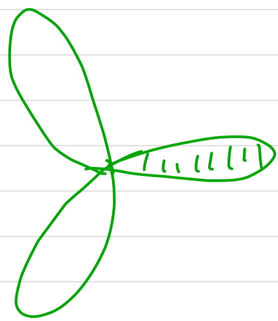
$$\text{Area}(R) = \int_{\partial R} x\,dy = \int_{\partial R} r\cos\theta \, d\, r\sin\theta = \int_{\partial R} r^2\cos^2\theta \, d\theta + r\cos\theta\sin\theta \, dr$$

$$\text{Area}(R) = -\int_{\partial R} y\,dx = -\int_{\partial R} r\sin\theta \, d\, r\cos\theta = \int_{\partial R} r^2\sin^2\theta \, d\theta - \int_{\partial R} r\sin\theta\cos\theta \, dr$$

$$2\text{Area}(R) = \int x\,dy - y\,dx = \int_{\partial R} r^2\, d\theta$$

$$\Rightarrow \text{Area}(R) = \frac{1}{2}\int_{\partial R} r^2\, d\theta$$

Q4 : Find the area of one leaf of the region bounded by $r = \cos 3\theta$.



Ans: $\text{Area} = \frac{1}{2}\int_{\partial R} r^2\, d\theta$

Step 1: $(r(t), \theta(t)) = (\cos 3t, t) \quad -\frac{\pi}{6} \le t \le \frac{\pi}{6}$

Step 2: $\text{Area}(R) = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 3t \, dt$

$$= \frac{1}{4}\int_{-\pi/6}^{\pi/6} 1 + \cos 6t \, dt$$

$$= \frac{\pi}{12}$$

Q5 Find the area enclosed by the curve

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = 1 \text{ and the x-axis and y-axis}$$

Ans: Let $u = x^{\frac{1}{2}}$, $v = y^{\frac{1}{2}}$, $\Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{vmatrix} 2u & \\ & 2v \end{vmatrix}\right| = |4uv|$

$x = u^2$, $y = v^2$,

$$\Rightarrow \text{Area} = 4\iint_{u+v\leq 1} |uv| \, du dv$$

$$= 4\int_0^1 \int_0^{1-v} uv \, du dv$$

$$= 2\int_0^1 v(1-v)^2 \, dv$$

$$= 2\int_0^1 v^3 - 2v^2 + v \, dv$$

$$= \frac{1}{6}$$

Q6 Let C be the intersection of

$$x^2+y^2+z^2=1 \quad \text{and} \quad z=0$$

Determine the orientation of C such that

$\int_C (y-z)\,dx + (z-x)\,dy + (x-y)\,dz$ is positive by writing it as an integral over the upper hemisphere.

Also find its value.

Ans:

$$\vec{F} = (y-z,\ z-x,\ x-y), \quad \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ y-z & z-x & x-y \end{vmatrix}$$

$$= -2\vec{i} - 2\vec{j} - 2\vec{k}$$

$\circlearrowleft$: $\vec{n}$ $\uparrow$

$\circlearrowright$: $\vec{n}$ $\downarrow$

Assume $\circlearrowleft$, let γ be a parametrization of the upper semi-sphere given by

$$\gamma(\theta,\varphi) = (\cos\varphi\cos\theta,\ \cos\varphi\sin\theta,\ \sin\theta)$$

Now $\iint_\Delta \nabla\times\vec{F}\cdot\vec{n}\,dA = \iint_\Delta (\nabla\times\vec{F})\cdot(\gamma_\theta\times\gamma_\varphi)\,d\theta\,d\varphi$ (remember, not $\gamma_\varphi\times\gamma_\theta$)

$$\gamma_\theta = (-\cos\varphi\sin\theta,\ \cos\varphi\cos\theta,\ 0)$$
$$\gamma_\varphi = (-\sin\varphi\cos\theta,\ -\sin\varphi\sin\theta,\ \cos\varphi)$$

$$\Rightarrow (\nabla\times\vec{F})\cdot(\gamma_\theta\times\gamma_\varphi) = \begin{vmatrix} -2 & -2 & -2 \\ -\cos\varphi\sin\theta & \cos\varphi\cos\theta & 0 \\ -\sin\varphi\cos\theta & -\sin\varphi\sin\theta & \cos\varphi \end{vmatrix}$$

$$\begin{vmatrix} -2 & -2 & -2 \\ -\cos\varphi\sin\theta & \cos\varphi\cos\theta & 0 \\ -\sin\varphi\cos\theta & -\sin\varphi\sin\theta & \cos\varphi \end{vmatrix}$$

$$= -2(\sin\varphi\cos\varphi\sin^2\theta + \sin\varphi\cos\varphi\cos^2\theta) + \cos\varphi(-2\cos\varphi\cos\theta - 2\cos\varphi\sin\theta)$$

$$= -2\sin\varphi\cos\varphi - 2\cos^2\varphi(\cos\theta + \sin\theta)$$

$$= -\sin 2\varphi - 2\cos^2\varphi(\cos\theta + \sin\theta)$$

So, $\iint_{\Delta} \nabla\times\vec{F}\cdot n = -\int_0^{\pi/2}\int_0^{2\pi} \sin2\varphi + 2\cos^2\varphi(\cos\theta+\sin\theta)\,d\theta\,d\varphi$

$$= -2\pi\int_0^{\pi/2} \sin2\varphi\,d\varphi$$

$$= \pi\left[\cos 2\varphi\right]_0^{\pi/2}$$

$$= -2\pi \quad < 0$$

Therefore, the orientation should be Ω,

and the value should be 2π.

Rmk 1: You don't have to use the upper-hemisphere, you can convert the line integral into a surface integral over any surface whose boundary is the curve.

e.g. the unit disc in the x-y plane:

$$\int_C \vec{F} = \int_{III} \nabla \vec{F} \cdot \vec{k} \, dA =$$

$$= \int_{IV} -2 \, dA$$

$$= -2\pi$$

Rmk 2: This line integral can be calculated directly using a parametrization,

e.g. $\gamma(\theta) = (\cos\theta, \sin\theta, 0)$

And $$\int_C \vec{F} = \int_0^{2\pi} \sin\theta \, d\cos\theta - \cos\theta \, d\sin\theta$$

$$= -\int_0^{2\pi} d\theta$$

$$2\pi$$

Q7 Let $\vec{F}_0 = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$

$\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathrm{curl}(\vec{F}) = 0$ on $\mathbb{R}^2 \setminus \{(0,0)\}$

Show that there exists $\beta \in \mathbb{R}$ with

$\oint_C \vec{F} - \beta \vec{F}_0 = 0$ for any simple closed loop on $\mathbb{R}^2 \setminus \{(0,0)\}$

Hints: Step 1: for a small circle C_ε around the origin, a) $\int_{C_\varepsilon} \vec{F}_0 = 2\pi$.

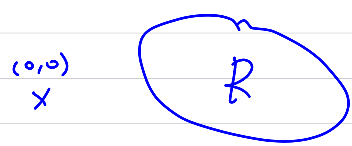
b) $\int_{C_\varepsilon} \vec{F}$ is independent of ε (by Green's thm)

So choose $\beta = \frac{1}{2\pi}\int_{C_\varepsilon} \vec{F}$

Step 2: Let C be a simple closed curve

R be the region bounded by C

Case 1 : $(0,0) \notin R$



Then $\int_C \vec{F} - \beta\vec{F}_0 = \int_R \nabla\times\vec{F} - \beta\nabla\times\vec{F}_0 = 0$

Case 2 : $(0,0) \in R$, choose a small circle C_ε in R

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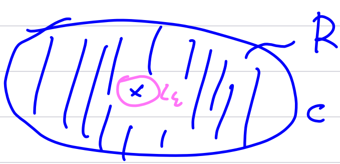

Then $\int_C \vec{F} - \beta \vec{F}_0 - \int_{C_\varepsilon} \vec{F} - \beta \vec{F} = \int_{R'} \nabla \times \vec{F} - \beta \nabla \times \vec{F} = 0$

$\Rightarrow \int_C \vec{F} - \beta \vec{F}_0 = \int_{C_\varepsilon} \vec{F} - \beta \cdot 2\pi = 0$